The Mathematical Experience

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With an Introduction by Gian-Carlo Rota

Non-Euclidean Geometry

Euclid
c. 300 B.C.

THE APPEARANCE on the mathematical scene a century and a half ago of non-Euclidean geometries was accompanied by considerable disbelief and shock. The existence of such geometries is now easily explained in a few sentences and will easily be understood. Any mathematical theory such as arithmetic, geometry, algebra, topology, etc., can be presented as an axiomatic scheme wherein consequences are deduced systematically and logically from the axioms. Such a logico-deductive scheme may be compared to a game and the axioms of the scheme to the rules of the game. Anyone who plays games knows that one can invent variations on given games and the consequences will be different. A non-Euclidean geometry is a geometry that is played with axioms that are different from those of Euclid.

Of course, this simple explanation violates the historical order. It borrows from a philosophy of mathematics which came about precisely as a result of the discovery of such geometries. For a fuller understanding of the matter, it is necessary to see what happened chronologically.

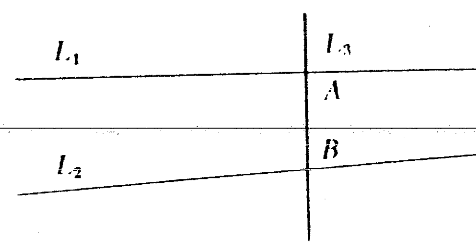
Since the Greeks, geometry has had a dual aspect. It is claimed to be an accurate description of the space in which we live and it is also an intellectual discipline, a deductive structure. These two aspects are now viewed as separate, but this was not always the case. The geometry of Euclid was based on a number of axioms and postulates of which we quote the first five postulates.

(The distinction between the words axiom and postulate is fuzzy. Modern mathematics uses the words almost interchangeably.)

1. A straight line may be drawn between any two points.
2. Any terminated straight line may be extended indefinitely.

3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.
5. If two straight lines lying in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.

In other words, referring to the figure,



if $\sphericalangle A + \sphericalangle B < 180°$ the lines L_1 and L_2 will intersect at some point on the right hand side of L_3.

The word axiom or postulate, in an earlier view, meant a self-evident or a universally recognized truth, a truth accepted without proof. Within deductive geometry, the axiom functions as a cornerstone on which further conclusions are based. Within descriptive geometry, the axiom functions as a true and accurate statement of the world of spatial experiences. The former view persists; the latter has had to give way.

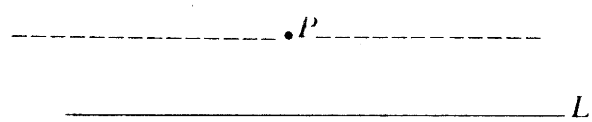
If one takes a look at Postulates 1, 2, 3, 4, they appear easy of statement and, indeed, self-evident. Postulate 5 is different. It is complicated to state and rather less self-evident. It seems to transcend direct physical experience. Postulate 5 is known as Euclid's Parallel Postulate or, more familiarly, in a friendly allusion to the Amendments of the United States Constitution, Euclid's Fifth. From the earliest times it attracted special attention.

The historical development of non-Euclidean geometry was a result of attempts to deal with this axiom. Notice, further, that although Euclid's Fifth is known as the parallel axiom, the word "parallel" does not occur in it. The word parallel is expanded in Euclid under Definition 23.

"Parallel straight lines are lines which being in the same plane and being produced indefinitely in both directions do not meet one another in either direction."

The reason for our calling Euclid's Fifth the parallel axiom is that it is totally equivalent to any of the following statements involving the word parallel:

1. If a straight line intersects one of two parallels, it will intersect the other.
2. Straight lines parallel to the same straight line are parallel to each other.
3. Two straight lines which intersect one another cannot be parallel to the same line.
4. Given, in a plane, a line L and a point P not on L. Then through P there exists one and only one line parallel to L.



Equivalence means that any of these statements plus the other axioms implies Euclid's Fifth, and vice versa.

Over the years, for reasons which are partly technical and partly aesthetic, the fourth formulation has come to be the standard formulation of Euclid's assertion of parallelism. It is known as the Playfair Axiom, after the Britisher John Playfair, 1748–1819.

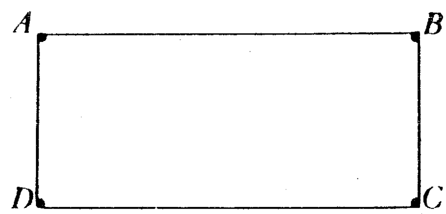
The early investigations with Euclid's Fifth tried to assuage the doubts of its validity by attempting to derive it logically from the other axioms, which seemed to be self-evident. It would then become a theorem and its status would be assured. These attempts failed, and for good reason—we now know that it cannot be so derived. This was established by 1868.

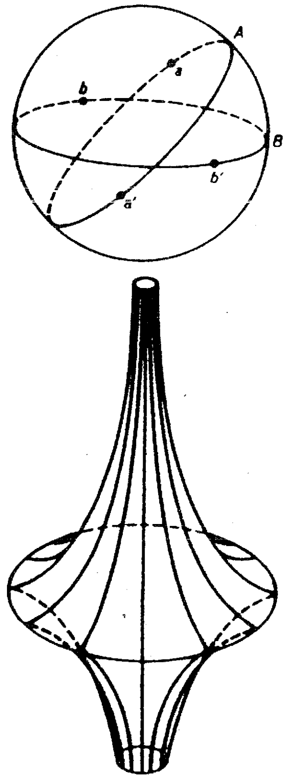
With the failure of direct methods, it was inevitable that mathematicians would turn to indirect methods. In such

Bernhard Riemann
1826–1866

an approach, one denies the Fifth and then tries to derive a contradiction.

Two notable investigators employing *reductio ad absurdam* were Girolamo Saccheri (1667–1733) and Johann Lambert (1728–1777). Both of these denied the Fifth.





On surface of a sphere "straight line" is interpreted to mean "great circle" (A and B at top). *Through any pair of diametrically opposite points* (aa' and bb') *there pass many great circles. If we interpret "point" to mean "point pair," then Euclid's first*

Saccheri works with a quadrilateral ABCD which has right angles at A and B and in which AD = BC. This is now known in axiomatic geometry as a Saccheri quadrilateral. It should be noted that within Euclidean geometry AD will be parallel to BC and this makes the angles at D and C both right angles. But Saccheri, not taking the Fifth, concludes that he really has three options:

1. The angles at C and D are both right angles.
2. They are both obtuse angles.
3. They are both acute angles.

Some of the conclusions from (2) and (3) are sufficiently startling to go against "intuition." At this point Saccheri throws in the towel and cries "contradiction."

Lambert, the bolder and more skillful of the two, lasts a few more rounds, and does not give up until he has discovered that within the new and hypothetical system he has devised one could prove the existence of an absolute unit of length. Since, he argues, there can be no absolute unit of length, the whole enterprise must have been fallacious.

It is not our intent here to trace the stream of discovery and its many tributary branches. Many mathematicians played a role—Gauss, Lobachevsky, Bolyai, and Riemann, to name the most important. The discovery was attended by many misunderstandings, doubts, misgivings. It seemed to be at the edge of madness. The birth pains were severe. Thus, e.g., Bolyai's father wrote to him "For God's sake, please give it up. Fear it no less than the sensual passions because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life."

It was found that there are not one but two non-Euclidean geometries. They currently go by the names of Lobachevskian (or hyperbolic) geometry and Riemannian (or elliptic) geometry. With respect to the Playfair Axiom, these two non-Euclidean geometries correspond to the axioms:

Lobachevsky: Given in a plane a line L and a point P not on L. Then there are at least two lines through P parallel to L.

Riemann: Given in a plane a line L and a point P not on L. Then there are *no* lines through P parallel to L.

The geometries, Euclidean, Lobachevskian, and Riemannian, carry with them three distinct sets of conclusions (theorems), which are worked out in detail in textbooks on the subject. There are different mensuration formulas, different projective aspects. A comparison between the three is most interesting. Some of the elementary differences are summed up in the table adjoined.

We give two more comparisons. The famous theorem of Pythagoras now has three forms.

Euclidean Geometry: $c^2 = a^2 + b^2$

Lobachevskian Geometry: $2(e^{c/k} + e^{-c/k}) = (e^{a/k} + e^{-a/k})(e^{b/k} + e^{-b/k})$ where k is a certain fixed constant and $e = 2.718\ldots$

Riemannian Geometry: Differential form: $ds^2 = \alpha dx^2 + 2\beta dxdy + \gamma dy^2$ where $\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}$ is positive definite.

The circumference C of a circle whose radius is r has the formulas

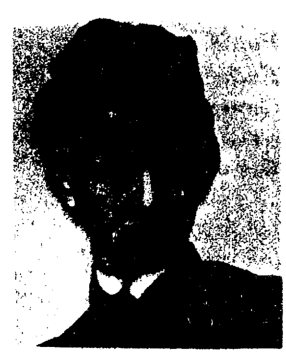
Euclidean Geometry: $C = 2\pi r$

Lobatchevskian Geometry: $C = \pi k(e^{r/k} - e^{-r/k})$.

The Riemannian formula for C is not expressible in simple terms.

It remains for us to discuss the logical consistency of non-Euclidean geometry. In order to do this we merely replace the word "line" everywhere by the phrase "great cir-

postulate is true. The second postulate is true if one allows the extended "straight line" to have a finite total length, or to retrace itself many times as it goes around the sphere. The third postulate is also true if one understands distance to be measured along great circles that can be retraced several times; here a "circle" means merely the set of points on the sphere at a given great-circle distance from a given point. The fourth postulate is likewise true. Playfair's postulate is false, because any two great circles intersect. Thus the sphere is a model of non-Euclidean geometry. So is the pseudo-sphere (bottom), if straight lines are interpreted as being the shortest curves connecting any two points on the surface. On the surface of the pseudosphere there are many "straight lines" that pass through a given point and do not cross a given straight line.



Janos Bolyai
1802–1860

	Euclidean	Lobachevskian	Riemannian	
Two distinct lines intersect in	at most one	at most one	one (single elliptic) two (double elliptic)	point points
Given line L and point P not on L, there exist	one and only one line	at least two lines	no lines	through P parallel to L
A line	is	is	is not	separated into two parts by a point
Parallel lines	are equidistant	are never equidistant	do not exist	
If a line intersects one of two parallel lines, it	must	may or may not	—	intersect the other
The valid Saccheri hypothesis is the	right angle	acute angle	obtuse angle	hypothesis
Two distinct lines perpendicular to the same line	are parallel	are parallel	intersect	
The angle sum of a triangle is	equal to	less than	greater than	180 degrees
The area of a triangle is	independent	proportional to the defect	proportional to the excess	of its angle sum
Two triangles with equal corresponding angles are	similar	congruent	congruent	

*Table Comparing Euclidean and Non-Euclidean Plane Geometry**

* *From Prenowitz and Jordan, Basic Concepts of Geometry.*

cle," a circle formed on the surface of a sphere by a plane passing through the center of the sphere. We now regard the axioms as statements about points and great circles on a given sphere. Moreover, we agree to identify each pair of diametrically opposite points on the sphere as a single point. If the reader prefers, he can imagine the axioms of non-Euclidean geometry rewritten, with the word "line" everywhere replaced by "great circle," the word "point" everywhere replaced by "point pair." Then it is evident that all the axioms are true, at least insofar as our ordinary notions about the surface of a sphere are true. In fact, from the axioms of Euclidean solid geometry one can easily prove as theorems that the surface of a sphere is a non-Euclidean surface in the sense we have just described. In other words, we now see that if the axioms of non-Euclidean geometry led to a contradiction, then so would the ordinary Euclidean geometry of spheres lead to a contradiction. Thus we have a *relative* proof of consistency; if Euclidean three-dimensional geometry is consistent, then so is non-Euclidean two dimensional geometry. We say that the surface of the Euclidean sphere is a model for the axioms of non-Euclidean geometry. (In the particular model we have used the parallel postulate fails because there are no parallel lines. It is also possible to construct a surface, the "pseudosphere," for which the parallel postulate is false because there is more than one line through a point parallel to a given line.)

Further Readings. See Bibliography

A. D. Alexandroff, Chapter 17; H. Eves and C. V. Newsom; E. B. Golos; M. J. Greenberg; M. Kline [1972], Chapter 36; W. Prenowitz and M. Jordan; C. E. Sjöstedt